Robust chaos in piecewise-linear maps

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Border-collision normal form

- Piecewise-linear maps arise when modeling systems with switches, thresholds and other abrupt events.
- In our project, we study the two-dimensional *border-collision normal form* (Nusse & Yorke, 1992), given by

$$f_\xi(x,y) = \begin{cases} \begin{bmatrix} \tau_L & 1 \\ -\delta_L & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}, & x \le 0, \\ \begin{bmatrix} \tau_R & 1 \\ -\delta_R & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}, & x \ge 0. \end{cases}$$

- Here $(x,y) \in \mathbb{R}^2$, and $\xi = (\tau_L, \delta_L, \tau_R, \delta_R) \in \mathbb{R}^4$ are the parameters.

Phase portrait of a chaotic attractor

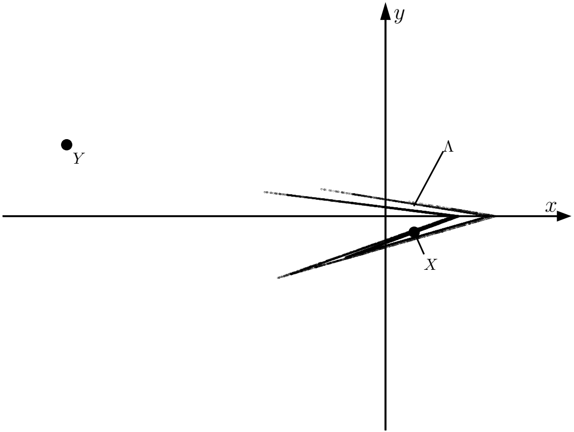


Figure: A sketch of the phase portrait of f_ξ with $\xi \in \Phi_{\mathrm{BYG}}$.

Renormalisation operator

- Renormalisation involves showing that, for some members of a family of maps, a higher iterate or induced map is conjugate to different member of this family of maps.
- Relevant dynamics arise in only two pieces of f_ξ^2,

$$
f_\xi^2(x,y) = \begin{cases} \begin{bmatrix} \tau_L\tau_R - \delta_L & \tau_R \\ -\delta_R\tau_L & -\delta_R \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \tau_R + 1 \\ -\delta_R \end{bmatrix}, & x \le 0, \\ \begin{bmatrix} \tau_R^2 - \delta_R & \tau_R \\ -\delta_R\tau_R & -\delta_R \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \tau_R + 1 \\ -\delta_R \end{bmatrix}, & x \ge 0. \end{cases}
$$

Renormalisation operator

- Now f_ξ^2 can be transformed to $f_{g(\xi)}$, where g is the *renormalisation operator* (Ghosh & Simpson, 2022.) $g : \mathbb{R}^4 \to \mathbb{R}^4$, given by

$$\left(\tilde{\tau}_L, \tilde{\delta}_L, \tilde{\tau}_R, \tilde{\delta}_R\right) = \left(\tau_R^2 - 2\delta_R, \delta_R^2, \tau_L\tau_R - \delta_L - \delta_R, \delta_L\delta_R\right) \tag{1}$$

- We perform a coordinate change to put f_ξ^2 in the normal form :

$$\begin{bmatrix} \tilde{x}' \\ \tilde{y}' \end{bmatrix} = \begin{cases} \begin{bmatrix} \tilde{\tau}_L & 1 \\ -\tilde{\delta}_L & 0 \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}, & \tilde{x} \leq 0, \\ \begin{bmatrix} \tilde{\tau}_R & 1 \\ -\tilde{\delta}_R & 0 \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}, & \tilde{x} \geq 0. \end{cases}$$

Results

- We consider the parameter region

$$\Phi = \left\{ \xi \in \mathbb{R}^4 \middle| \tau_L > \delta_L + 1, \delta_L > 0, \tau_R < -(\delta_R + 1), \delta_R > 0 \right\}.$$

- Let

$$\phi^+(\xi) = \zeta_0 = \delta_R - (\tau_R + \delta_L + \delta_R - (1 + \tau_R)\lambda_L^u)\lambda_L^u.$$

- The stable and the unstable manifolds of the fixed point Y intersect if and only if $\phi^+(\xi) \leq 0$.
- The attractor is often destroyed at $\phi^+(\xi) = 0$ which is a homoclinic bifurcation (Banerjee, Yorke & Grebogi, 1998), and thus focused their attention on the region

$$\Phi_{\mathrm{BYG}} = \left\{ \xi \in \Phi \middle| \phi^+(\xi) > 0 \right\}.$$

Results

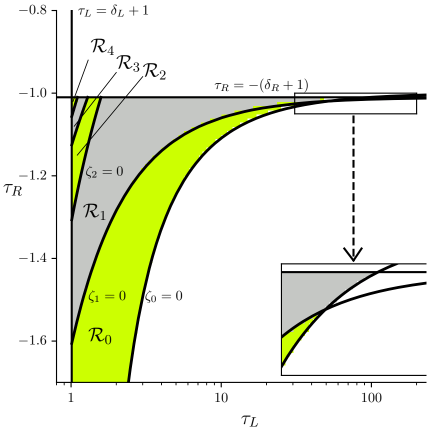


Figure: The sketch of two-dimensional cross-section of Φ_{BYG} when $\delta_L = \delta_R = 0.01$.

Results

Theorem (Ghosh & Simpson, 2022)

The $\mathcal{R}_n$ *are non-empty, mutually disjoint, and converge to the fixed point* $(1, 0, -1, 0)$ *as* $n \to \infty$. *Moreover,*

$$\Phi_{\mathrm{BYG}} \subset \bigcup_{n=0}^{\infty} \mathcal{R}_n.$$

Let,

$$\Lambda(\xi) = \mathrm{cl}(W^u(X)).$$

Theorem (Ghosh & Simpson, 2022)

For the map f_ξ *with any* $\xi \in \mathcal{R}_0$, $\Lambda(\xi)$ *is bounded, connected, and invariant. Moreover,* $\Lambda(\xi)$ *is chaotic (positive Lyapunov exponent).*

Results

Theorem (Ghosh & Simpson, 2022)

For any $\xi \in \mathcal{R}_n$ where $n \geq 0$, $g^n(\xi) \in \mathcal{R}_0$ and there exist mutually disjoint sets $S_0, S_1, \ldots, S_{2^n-1} \subset \mathbb{R}^2$ such that $f_\xi(S_i) = S_{(i+1) \text{ mod } 2^n}$ and

$$f_\xi^{2^n}|_{S_i} \text{ is affinely conjugate to } f_{g^n(\xi)}|_{\Lambda(g^n(\xi))}$$

for each $i \in \{0, 1, \ldots, 2^n - 1\}$. Moreover,

$$\bigcup_{i=0}^{2^n-1} S_i = \mathrm{cl}(W^u(\gamma_n)),$$

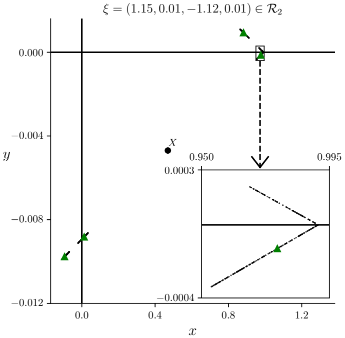
where γ_n is a saddle-type periodic solution of our map f_ξ having the symbolic itinerary $\mathcal{F}^n(R)$ given by Table 1.

Results

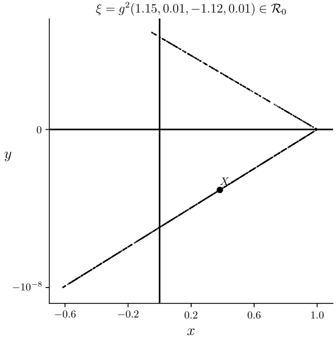
n	$\mathcal{F}^n(\mathcal{W})$
0	R
1	LR
2	$RRLR$
3	$LRLRRRLR$
4	$RRLRRRLRLRLRRRLR$

Table: The first 5 words in the sequence generated by repeatedly applying the substitution rule $(L, R) \mapsto (RR, LR)$ to $\mathcal{W} = R$.

Results

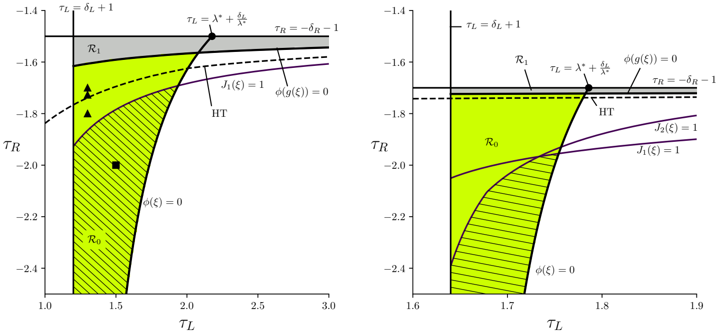


$\xi = (1.15, 0.01, -1.12, 0.01) \in \mathcal{R}_2$
0.000
−0.004
−0.008
−0.012
y
X
0.950
0.995
0.0003
−0.0004
0.0
0.4
0.8
1.2
x



$\xi = g^2(1.15, 0.01, -1.12, 0.01) \in \mathcal{R}_0$
0
y
X
-10^{-8}
−0.6
−0.2
0.2
0.6
1.0
x

Devaney Chaos



(a) $\delta_L = 0.2$, $\delta_R = 0.5$ (b) $\delta_L = 0.64$ and $\delta_R = 0.7$

Devaney Chaos

Theorem (Ghosh & Simpson, 2022)

Let $\xi \in \Phi_{\mathrm{BYG}}$ and suppose $J_1(\xi) > 1$ and $\lambda_L^s + |\lambda_R^s| < 1$. Then $W^s(X)$ is dense in a triangular region containing Λ.

Theorem (Ghosh & Simpson, 2022)

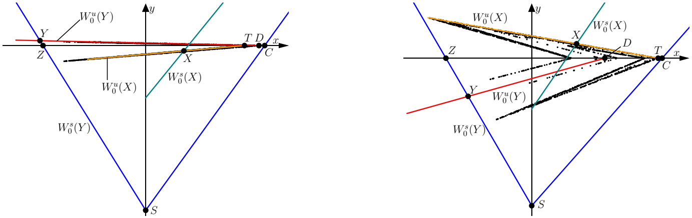
Let $\xi \in \Phi_{\mathrm{BYG}}$ and suppose $J_1(\xi) > 1$ and $J_2(\xi) < 1$. Then, f_ξ is chaotic in the sense of Devaney on Λ.

Generalised parameter region

Now we consider the more generalised parameter region considering the orientation-reversing and non-invertible cases,

$$\Phi = \left\{ \xi \in \mathbb{R}^4 \;\middle|\; \tau_L > |\delta_L + 1|, \tau_R < -|\delta_R + 1| \right\}.$$

Typical phase portraits

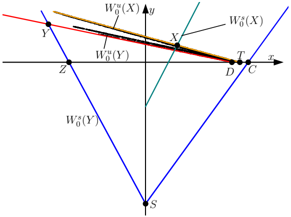


(a) $\delta_L > 0, \delta_R > 0$

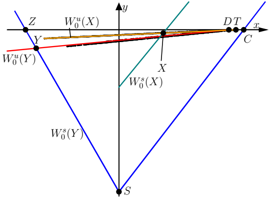
(b) $\delta_L < 0, \delta_R < 0$

Figure: Typical phase portraits of the chaotic attractor for the invertible case ($\delta_L \delta_R > 0$).

Typical phase portraits



(a) $\delta_L > 0, \delta_R < 0$



(b) $\delta_L < 0, \delta_R > 0$

Figure: Typical phase portraits of the chaotic attractor for the non-invertible case $(\delta_L \delta_R < 0)$.

Invariant expanding cones

Chaos in Φ_{BYG} can be proved by constructing an invariant expanding cone in tangent space (Glendinning & Simpson, 2021). We have extended this to Φ.

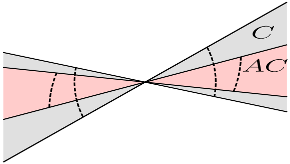


Figure: A sketch of an invariant expanding cone C and its image $AC = \{Av | v \in C\}$, given $A \in \mathbb{R}^{2\times 2}$.

Robust Chaos in a generalised setting

Theorem (Ghosh, McLachlan, & Simpson, 2023)

For any $\xi \in \Phi_{\mathrm{trap}} \cap \Phi_{\mathrm{cone}}$, *the normal form* f_ξ *has a topological attractor with a positive Lyapunov exponent.*

Robust Chaos in a generalised setting

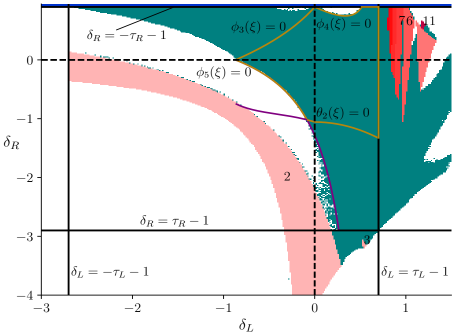


Figure: A 2D slice of $\Phi_{\text{trap}} \cap \Phi_{\text{cone}} \subset \mathbb{R}^4$.

The orientation-reversing case

- Let

$$\Phi^{(2)} = \{\xi \in \Phi \mid \delta_L < 0, \delta_R < 0\},$$

be the subset of Φ for which the BCNF is orientation-reversing.

- The attractor Λ which is again a closure of the unstable manifold of X faces a crisis at $\zeta_0^{(2)} = 0$ where

$$\zeta_0^{(2)} = \phi^-(\xi) = \delta_R - (\delta_R + \tau_R - (1 + \lambda_R^u)\lambda_L^u)\lambda_L^u\,.$$

The orientation-reversing case

- Now, $\xi \in \Phi^{(2)}$ implies $g(\xi) \in \Phi^{(1)}$, so we again use the preimages of $\phi^+(\xi) = 0$ under g to define the region boundaries: Specifically we let

$$\mathcal{R}_0^{(2)} = \left\{ \xi \in \Phi^{(2)} \,\middle|\, \phi^-(\xi) > 0, \phi^+(g(\xi)) \leq 0, \alpha(\xi) < 0 \right\},$$
$$\mathcal{R}_n^{(2)} = \left\{ \xi \in \Phi^{(2)} \,\middle|\, \phi^+\left(g^n(\xi)\right) > 0, \phi^+\left(g^{n+1}(\xi)\right) \leq 0, \alpha(\xi) < 0 \right\}, \qquad \text{for all } n \geq 1.$$

  where

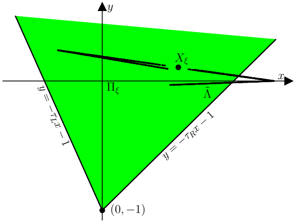
$$\alpha(\xi) = \tau_L \tau_R + (\delta_L - 1)(\delta_R - 1).$$

- This brings us to the proposition

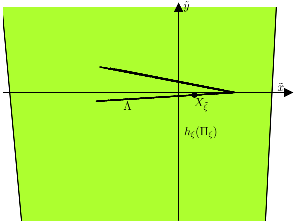
Proposition (Ghosh, McLachlan, & Simpson, 2024)

If $\xi \in \mathcal{R}_n^{(2)}$ *with* $n \geq 1$*, then* $g(\xi) \in \mathcal{R}_{n-1}^{(1)}$.

The orientation-reversing case



(a) $\xi = \xi_{\text{ex}}^{(2)} \in \mathcal{R}_1^{(2)}$



(b) $\xi = g(\xi_{\text{ex}}^{(2)}) \in \mathcal{R}_0^{(1)}$

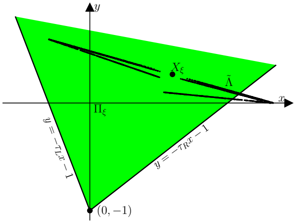
The non-invertible case $\delta_L > 0, \delta_R < 0$

- Let
$$\Phi^{(3)} = \{\xi \in \Phi \mid \delta_L > 0, \delta_R < 0\},$$
meaning the map is invertible.
- In this region an attractor can be destroyed by crossing the homoclinic bifurcation $\phi^+(\xi) = 0$ or the heteroclinic bifurcation $\phi^-(\xi) = 0$.
- we define
$$\phi_{\min}(\xi) = \min[\phi^+(\xi), \phi^-(\xi)].$$
and
$$\mathcal{R}_n^{(3)} = \left\{ \xi \in \Phi^{(3)} \,\middle|\, \phi_{\min}\left(g^n(\xi)\right) > 0,\ \phi_{\min}\left(g^{n+1}(\xi)\right) \le 0,\ \alpha(\xi) < 0 \right\},$$
for all $n \ge 0$.

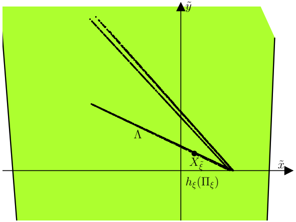
The non-invertible case $\delta_L > 0, \delta_R < 0$

- This brings us to a new proposition:

Proposition (Ghosh, McLachlan, & Simpson, 2024)
If $\xi \in \mathcal{R}_n^{(3)}$ with $n \geq 1$, then $g(\xi) \in \mathcal{R}_{n-1}^{(3)}$.



(a) $\xi = \xi_{\mathrm{ex}}^{(3)} \in \mathcal{R}_1^{(3)}$



(b) $\xi = g(\xi_{\mathrm{ex}}^{(3)}) \in \mathcal{R}_0^{(3)}$

The non-invertible case $\delta_L < 0, \delta_R > 0$

- It remains for us to consider

$$\Phi^{(4)} = \{\xi \in \Phi \mid \delta_L < 0, \delta_R > 0\},$$

where the BCNF is again non-invertible.

- In this region the attractor is usually destroyed before the boundaries $\phi^+(\xi) = 0$ and $\phi^-(\xi) = 0$ in a heteroclinic bifurcation that cannot be characterised by an explicit condition on the parameter values.
- Despite the extra complexities in $\Phi^{(4)}$ it still appears that renormalisation is helpful for explaining the bifurcation structure. Let

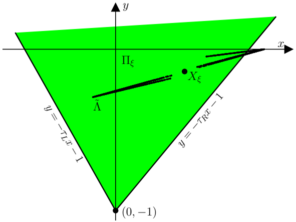
$$\mathcal{R}_0^{(4)} = \left\{\xi \in \Phi^{(4)} \;\middle|\; \phi_{\min}(\xi) > 0,\, \phi_{\min}(g(\xi)) \le 0,\, \alpha(\xi) < 0\right\}.$$

$$\mathcal{R}_n^{(4)} = \left\{\xi \in \Phi^{(4)} \;\middle|\; \phi_{\min}(g^n(\xi)) > 0,\, \phi_{\min}(g^{n+1}(\xi)) \le 0,\, \alpha(\xi) < 0, \alpha(g(\xi)) < 0\right\}. \tag{2}$$

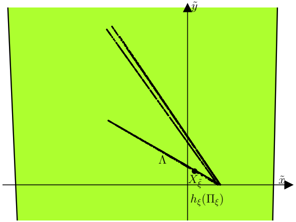
The non-invertible case $\delta_L < 0, \delta_R > 0$

- This brings us to the new propostion:

Proposition (Ghosh, McLachlan, & Simpson, 2024)
If $\xi \in \mathcal{R}_n^{(4)}$ with $n \geq 1$, then $g(\xi) \in \mathcal{R}_{n-1}^{(3)}$.



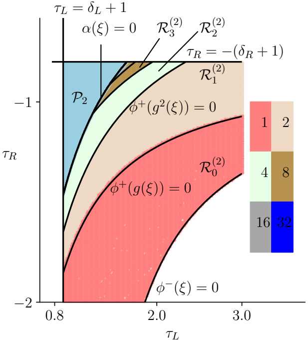
(a) $\xi = \xi_{\text{ex}}^{(4)} \in \mathcal{R}_1^{(4)}$



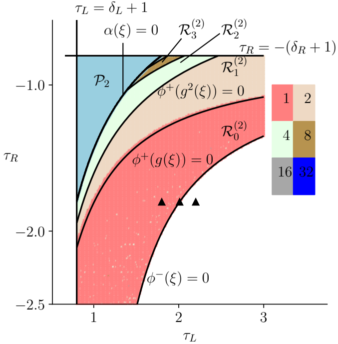
(b) $\xi = g(\xi_{\text{ex}}^{(4)}) \in \mathcal{R}_0^{(3)}$

Numerics

- Numerical verification using Eckstein's greatest common divisor algorithm (Eckstein, 2006), described by Avrutin *et al*, 2007.

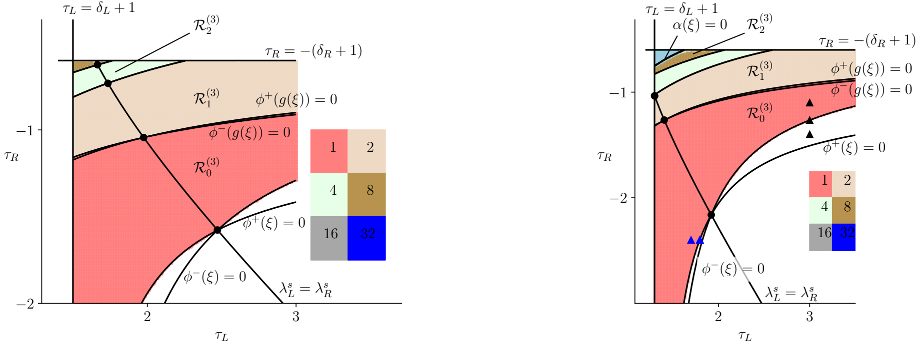


(a) $\delta_L = -0.1, \delta_R = -0.2.$



(b) $\delta_L = -0.2, \delta_R = -0.2.$

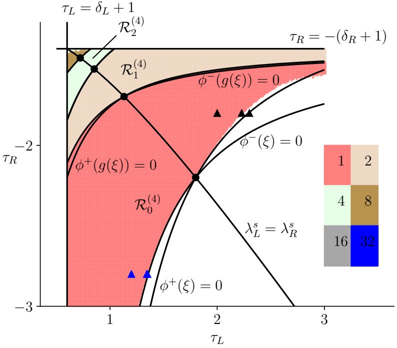
Numerics



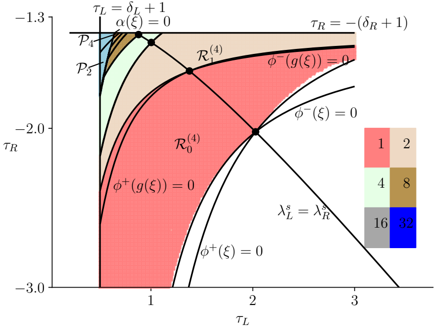
(a) $\delta_L = 0.5, \delta_R = -0.4.$

(b) $\delta_L = 0.3, \delta_R = -0.4.$

Numerics

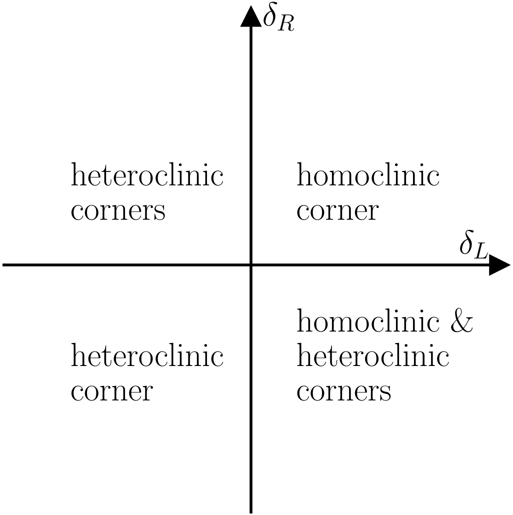


(a) $\delta_L = -0.4, \delta_R = 0.4.$



(b) $\delta_L = -0.5, \delta_R = 0.4.$

Numerics


δ_R
heteroclinic corners
homoclinic corner
δ_L
heteroclinic corner
homoclinic & heteroclinic corners

Higher-dimensional setting

- Let $n \geq 2$. Suppose $\alpha > 1$ is an eigenvalue of A_L, and $-\beta < -1$ of A_R with multiplicity one, and all other eigenvalues of A_L and A_R have modulus at most $0 < r < 1$.

Theorem (Ghosh & Simpson, 2024)

Holding the above assumption and

$$r(n-1) < \frac{3}{7}\left(1 - \frac{1}{\alpha}\right), \qquad r(n-1) < \frac{3}{7}\left(1 - \frac{1}{\beta}\right),$$

$$r(n-1) < \frac{1}{10}\left(\frac{1}{\alpha} + \frac{1}{\beta} - 1\right),$$

then f has a topological attractor with a positive Lyapunov exponent.

Higher-dimensional setting

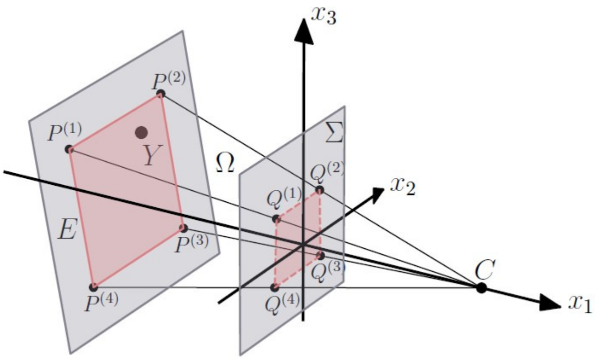


Figure: The construction of a forward invariant region Ω for $n = 3$.

Higher-dimensional setting

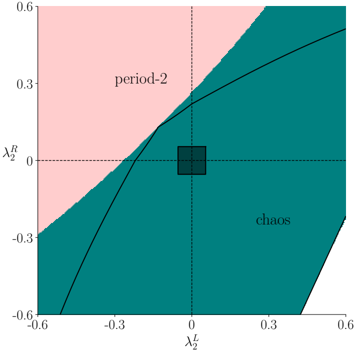


Figure: Robust chaos parameter region for the two-dimensional map, with our higher-dimensional construction portrayed on top of it. We chose $n = 2$ for simplicity.

Future Directions

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- Maps with multiple directions of instability should be just as relevant, giving the possibility of so-called wild chaos, and it remains to treat these scenarios.

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- It would be interesting to see if renormalisation schemes based on other symbolic substitution rules can be used to explain parameter regimes where the BCNF has attractors with other numbers of components, e.g. three components.
- Maps with multiple directions of instability should be just as relevant, giving the possibility of so-called wild chaos, and it remains to treat these scenarios.
- As one application I want to apply n–dimensional construction as the key space for an encryption scheme.

The End

Thank you! Questions?